Sec. 14.2:
Limits and Continuity

What We Will Go Over In Section 14.2

1. Limits of Functions of 2 Variables
2. Continuity of Functions of 2 Variables
3. Limits and Continuity of Functions of 3 or More Variables

What is a limit? For a 1-variable (Calc. 1) function…

1) $\lim_{x \to a} f(x) = L$ means …

if you plug in numbers closer and closer to *a* (on either side of *a* but not *a* itself!) into the function, the outputs will get closer and closer to *L*.

2) $\lim_{x \to a^-} f(x) = L$ means …

if you plug in numbers closer and closer to *a* but less than *a* into the function, the outputs will get closer and closer to *L*.

3) $\lim_{x \to a^+} f(x) = L$ means …

if you plug in numbers closer and closer to *a* but greater than *a* into the function, the outputs will get closer and closer to *L*.

What is a limit? For a 1-variable (Calc. 1) function…

For a 1-variable function, there's only 2 ways to approach a number *a*

Draw pic on board with a number line for the input and another number line for the output

3

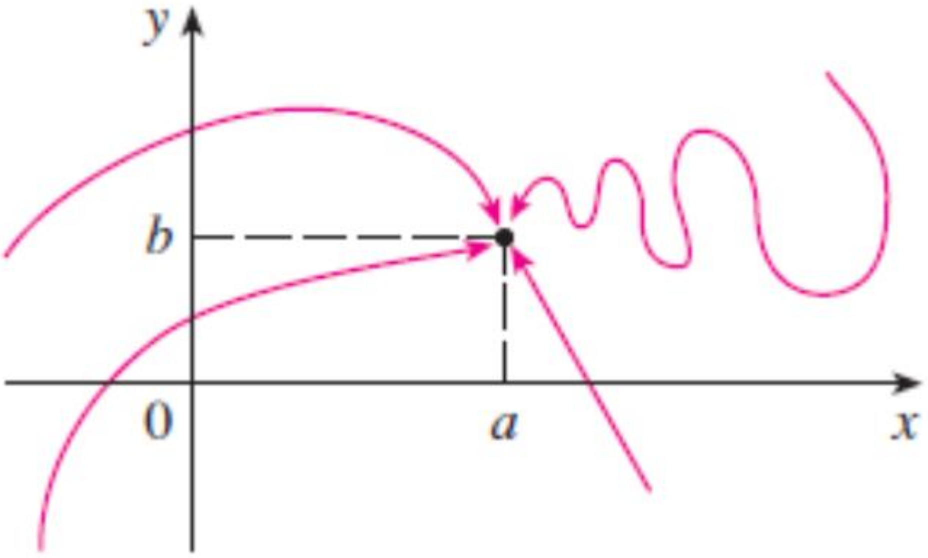
$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

What is a limit? For a 2-variable function…

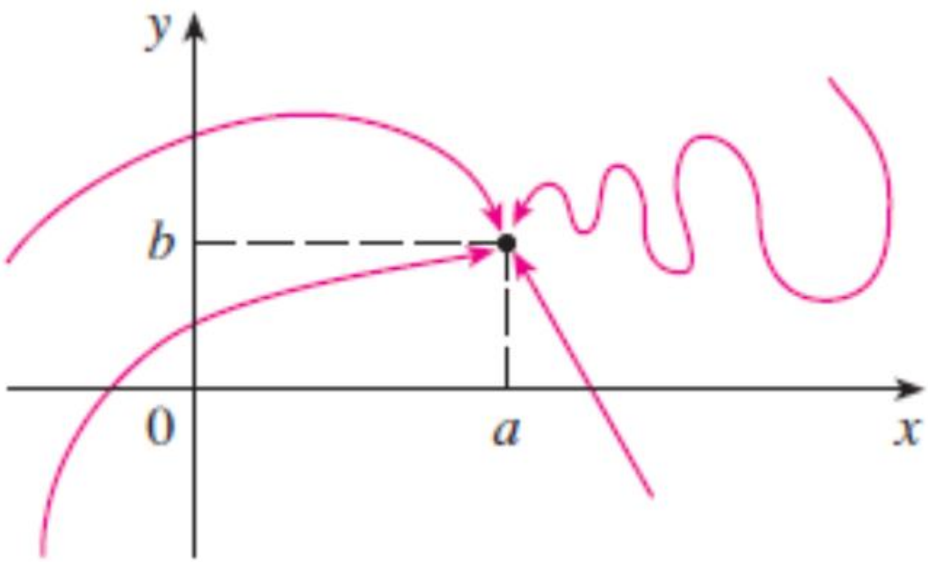
For a 2-variable function there are many ways to approach a point (a, b)



<u>What is a limit?</u> For a 2-variable function…

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \qquad \text{means} \ldots$$

if you plug ordered pairs (x, y) that are getting closer and closer to (a, b) (along any curve heading towards (a, b) but not (a, b) itself!) into the function, the outputs will get closer and closer to L.



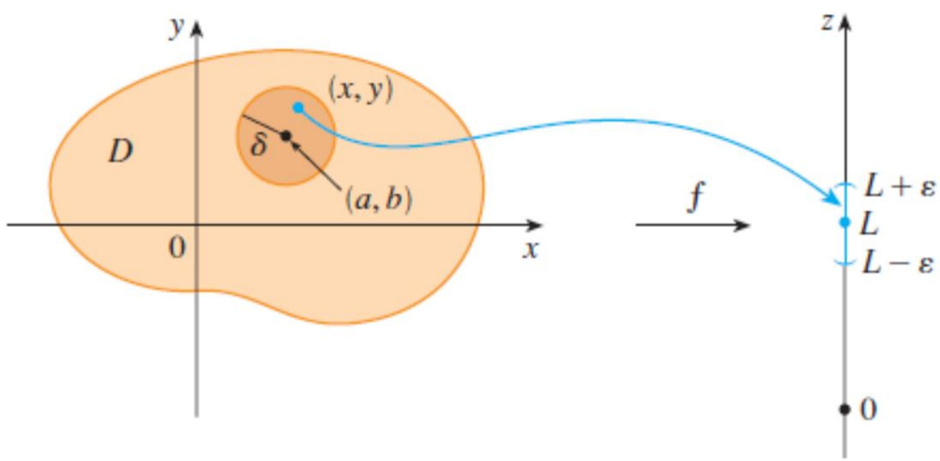
Draw pic of outputs on board

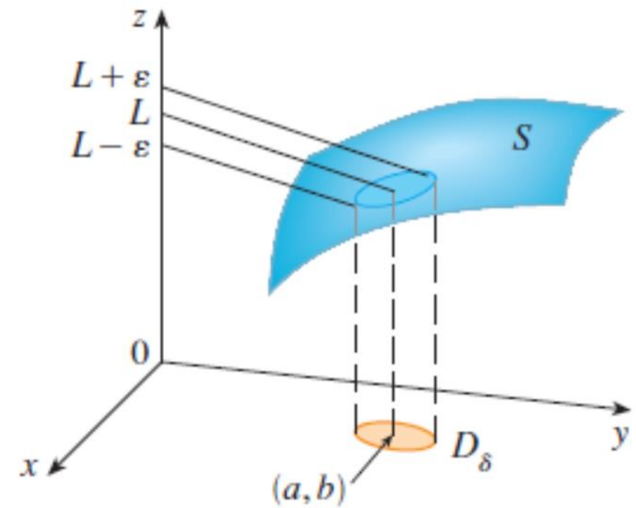
What is a limit? For a 2-variable function…

Formal definition of a limit of a 2-variable function…

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \qquad \text{means …}$$

For every $\epsilon > 0$, there is a $\delta > 0$ such that
if $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, then $|f(x,y) - L| < \epsilon$





What is a limit? For a 2-variable function…

Formal definition of a limit of a 2-variable function…

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \qquad \text{means } \dots$$

For every $\epsilon > 0$, there is a $\delta > 0$ such that
if $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, then $|f(x,y) - L| < \epsilon$

Draw pic on board

Even though this is the real definition, we won't use the $\epsilon - \delta$ definition to find limits in this class.

1. Limits of Functions of 2 Variables

To show that the limit of a 2-variable function does not exist…

1. Choose a curve C_1 that passes through (a, b) in the *xy*-plane
2. Parametrize the curve
3. Find the limit as points from this curve that are headed towards (a, b) are plugged into the function (this will be a 1-variable calc. 1 type of limit)
4. Repeat with a different curves until you find a curve C_2 that gives you a different limit

Then the limit does not exist!

1. Limits of Functions of 2 Variables

Ex 1: Show that $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

1. Limits of Functions of 2 Variables

Ex 2: If $f(x,y) = \dfrac{xy}{x^2 + y^2}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

1. Limits of Functions of 2 Variables

Ex 3: If $f(x,y) = \dfrac{xy^2}{x^2+y^4}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

1. Limits of Functions of 2 Variables

To show that the limit of a 2-variable function exists…

1. Switch to polar coordinates
2. Use the equations

$$x = rcos\theta \quad , \quad y = rsin\theta \quad , \quad x^2 + y^2 = r^2$$

   to turn the function into a function of r and θ only
3. Replace $(x, y) \to (0,0)$ with $r \to 0^+$
4. If only r appears in the new limit, then do this Calc. 1 limit
5. If θ appears in the limit, you may need to use the squeeze theorem to get rid of it
6. If (x, y) is approaching a point other than $(0,0)$, you can do a shift and make it approach $(0,0)$

1. Limits of Functions of 2 Variables

Ex 4: Find $\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)\ln(x^2+y^2)$ if it exists.

1. Limits of Functions of 2 Variables

Ex 5: Find $\lim_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2}$ if it exists.

1. Limits of Functions of 2 Variables

Ex 6: Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^5 + y^5}{x^2 + y^2}$ if it exists.

2. Continuity of Functions of 2 Variables

There are a few ways to think of continuity…

One way

$f(x, y)$ is continuous at (a, b) if

1. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists,

2. $f(a, b)$ exists, AND

3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$

2. Continuity of Functions of 2 Variables

There are a few ways to think of continuity…

Second way

$f(x, y)$ is continuous at (a, b) if when finding

$\lim_{(x,y)\to(a,b)} f(x, y)$, you can plug in (a, b) to get the

correct answer

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

2. Continuity of Functions of 2 Variables

Ex 7: Find $\lim_{(x,y)\to(1,2)} \left(x^2y^3 - x^3y^2 + 3x + 2y\right)$

2. Continuity of Functions of 2 Variables

Ex 8: Where is the function $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

2. Continuity of Functions of 2 Variables

Ex 9: Is $f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$

continuous on $\mathbb{R}^2$?

2. Continuity of Functions of 2 Variables

Ex 10: Where is the function $h(x, y) = \arctan(y/x)$

continuous?